STUDENT NUMBER: ____________

THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2006

MATHEMATICS

EXTENSION 1

Teacher Responsible: Mrs P Singh

General Instructions:

- Reading time – 5 minutes
- Working time – 2 hours
- This paper contains 7 questions.
- ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- Start each question in a new booklet.
- A table of standard integrals is supplied at the back of this paper.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

Question One 12 marks

(a) Differentiate with respect to x:

(i) $\cos^3 x$ 1

(ii) $e^x \log_e(5x)$ 2

(b) Solve the inequality $\frac{3}{x+4} \geq 1$ 3

(c) A is the point $(1, 2)$ and P is the point $(4, 5)$. P divides the interval AB internally in the ratio 3:5.

Find the co-ordinates of B. 3

(d) Find the exact value of $\int_0^3 \frac{2\,dx}{3+x^2}$. 3

Examination cont'd over page …/

Question Two

12 marks

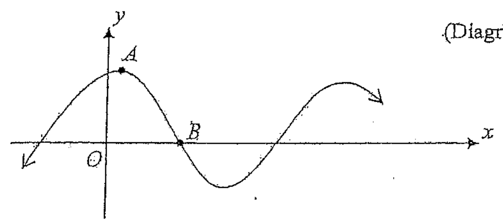
(a) Suppose that the cubic $F(x) = x^3 + ax^2 + bx + c$ has a relative minimum at $x = \alpha$ and a relative maximum at $x = \beta$.

(i) By examining the zeroes of $F'(x)$, prove that $\alpha + \beta = -\frac{2}{3}a$. 2

(ii) Deduce that a point of inflexion occurs at $x = \frac{1}{2}(\alpha + \beta)$. 2

(b) (i) Express $\sqrt{3}\ \sin x + \cos x$ in the form $R\sin(x+\alpha)$ where $R > 0$. 2

(ii)



(Diagram not to scale)

The graph of $y = \sqrt{3}\ \sin x + \cos x$ is shown above. A is a turning point.

Write down the co-ordinates of A and B. 4

(c) Find the size of the acute angle between the lines $3x - 2y + 1 = 0$ and $5x + y - 7 = 0$. 2

Examination cont'd over page ..../

Question Three

12 marks

(a) (i) Prove that $\dfrac{1-\cos 2x}{1+\cos 2x} = \sec^2 x - 1$. 3

(ii) Hence evaluate $\displaystyle\int_0^{\pi/4} \frac{1-\cos 2x}{1+\cos 2x}\,dx$. 3

(b) Sand is in a heap in the shape of a right circular cone whose height equals the radius of the base. More sand is added at the rate of 8 m^3/min. (Volume of cone $= \frac{1}{3}\pi r^2 h$).

When the heap is 10 m high, find:

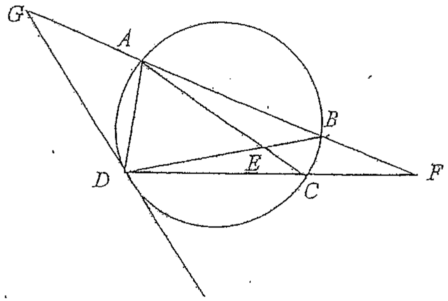
(i) the rate at which the height is increasing (in terms of π), 3

(ii) the rate at which the area of the base is increasing. 3

Examination cont'd over page .../

Question Four 12 marks

(a)



(Diagram not to scale)

DG is a tangent to the circle at D.

$GABF$ and DCF are straight lines.

(i) Copy the diagram into you answer booklet.

(ii) By letting $\angle GDA = x$, prove that $2\angle ADG = \angle BEC + \angle BFC$. 4

(b) (i) Find the points of intersection of the graphs $y = 10 - x^2$ and $y = \dfrac{9}{x^2}$. 3

(ii) Sketch these two graphs on the same set of axes. 3

(iii) Using your graphs, or otherwise, solve $10 - x^2 > \dfrac{9}{x^2}$. 2

Examination cont'd over page .../

Question Five 12 marks

(a) Prove by mathematical induction that $7^n - 1$ is divisible by 3 for any positive integer n. 4

(b) Consider the function given by $f(x) = \sin^{-1}x + \cos^{-1}x,\ 0 \le x \le 1$.

(i) Evaluate $f(0)$. 1

(ii) Find $f'(x)$. 1

(iii) Sketch the graph of $y = f(x)$. 2

(c) If $x = \tan\theta + \sec\theta$, use the t-formulae (i.e. $t = \tan\frac{\theta}{2}$) to show that $\dfrac{x^2 - 1}{x^2 + 1} = \sin\theta$. 4

Examination cont'd over page .../

Question Six

12 marks

(a) A loan of \$50,000 is to be repaid over 20 years by equal monthly instalments. Interest is calculated monthly on the balance owing at a rate of 12% p.a.

(i) Write down an expression for A_1, the amount owing at the end of the first month. 1

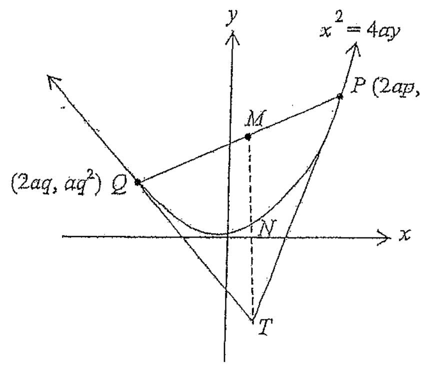
(ii) Show that A_n, the amount owing at the end of n months, is given by

$$A_n = 50000(1.01)^n - 100M(1.01^n - 1)$$

And hence find the monthly repayment, M. 2

(iii) The borrower decides to pay twice the minimum amount each month. i.e. \$2M each month. How long would it then take to pay off the loan? 3

(b) P and Q are points on the parabola $x^2 = 4ay$ with parameters p and q. The tangents at P and Q meet at T. M is the midpoint of PQ and MT is drawn to cut the parabola at N.



(Diagram not to scale)

(i) Prove that MT is parallel to the axis of the parabola. 3

(ii) Prove that N is the midpoint of MT. 3

Examination cont'd over page .../

Question Seven

12 marks

(a). The 'half-life' of a radio active substance is the time it takes a given amount of the substance to lose one-half of its mass. It is given that the 'half-life' of radium is 1700 years.

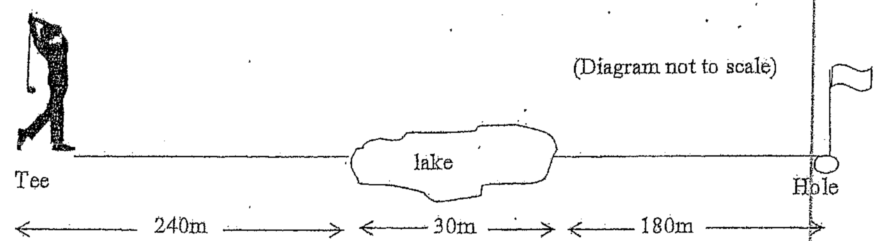
Assuming that radium decays according to the law $M = M_0e^{-kt}$ where M measures mass, t measures time and M_0 and k are constants, find how long it would take for an amount of radium to lose 80% of its mass. (Answer to nearest 100 years). 5

(b) A projectile is fired from the origin O at a speed v ms^{-1} and an angle of elevation of α (where $\alpha \neq \pi/2$).

You may assume that $x = vt\cos\alpha$ and $y = vt\sin\alpha - \frac{1}{2}gt^2$ where x and y are horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing and g is the acceleration due to gravity.

Show that the horizontal range of the projectile is $\dfrac{v^2\sin 2\alpha}{g}$ m. 4

(c)



Vijay Singh hits a golf ball at an initial speed of 50 ms^{-1} from a tee on a level fairway towards a hole 450 m away. There is a lake 30 m wide which is situated 240 m from the tee. Vijay knows that if the ball lands within 1 m either side of the lake, it will roll into the lake. At what angle should he attempt to hit the ball in order for it to travel the furthest possible distance towards the hole, without landing in the lake? (Take $g = 10$ms^{-2} and answer to nearest degree. You may use the result from (b) if you wish). 3

END OF EXAMINATION

Solution 2006 (Trial HSC Ext 1

Question 1

a) (i) $y = \cos^3 x$

$y' = 3\cos^2 x \cdot (-\sin x)$

$= -3\sin x \cos^2 x$ (1)

(ii) $y = e^x \log 5x$

$y' = e^x \log 5x + e^x \cdot \frac{1}{x}$

$= e^x\left(\log 5x + \frac{1}{x}\right)$ (2)

(b) $\frac{3}{x+4} \geq 1 \qquad x \neq -4$

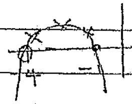
$3(x+4) \geq (x+4)^2$

$3(x+4) - (x+4)^2 \geq 0$

$(x+4)(3 - x - 4) \geq 0$

$(x+4) \cdot (-x-1) \geq 0$ (3)

$-4 < x \leq -1$



(c) $A(1,2) \qquad P\left(\frac{4}{}, \frac{5}{}\right)$

$B(x, y)$

$4 = \frac{3x+5}{8}$

$32 = 3x+5$

$x = 9$

$5 = \frac{3y+10}{8}$

$40 = 3y+10$

$y = 10$ (3)

$\therefore B$ ~~(11, 6)~~

$B(9, 10)$

d) $\int_1^4 \frac{dx}{x+\sqrt{x}} \qquad u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$x = 4,\ u = 2$

$x = 1,\ u = 1$

$\therefore \int_1^4 \frac{dx}{x+\sqrt{x}}$

$= 2\int_1^4 \frac{1}{\sqrt{x}+1} \cdot \frac{1}{2\sqrt{x}}\, dx$

$= 2\int_1^2 \frac{1}{u+1}\, du$

$= 2\left[\ln(u+1)\right]_1^2$

$= 2(\ln 3 - \ln 2)$

$= 2\ln\frac{3}{2}$

(e) $\int_0^3 \frac{2\,dx}{3+x^2}$

$= 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^3$

$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{3}{\sqrt{3}} - \tan^{-1} 0\right)$

$= \frac{2\pi}{3\sqrt{3}}$ (3)

Question 2

(a) (i) $F'(x) = 3x^2 + 2ax + b$

$\alpha + \beta = \frac{-2a}{3}$ (2)

(ii) $F''(x) = 6x + 2a$

$6x = -2a \quad (F''(x) = 0)$

$x = -\frac{1}{3}a$

$-\frac{2a}{3} = (\alpha+\beta)$

$\therefore -\frac{a}{3} = \frac{(\alpha+\beta)}{2}$ (2)

as reqd.

(b)(i) $\sqrt{3}\sin x + \cos x$

$= R\sin(x+\alpha), \quad R > 0$

$R = \sqrt{3+1}$

$= 2$

$2\sin(x+\alpha)$

$= 2\sin x\cos\alpha + 2\cos x\sin\alpha$

$2\cos\alpha = \sqrt{3}$

$2\sin\alpha = 1$

$\therefore \tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$ (3)

$\therefore \sqrt{3}\sin x + \cos x$

$= 2\sin\left(x + \frac{\pi}{6}\right)$

(ii)

b(ii) At max TP,

$\sin\left(x + \frac{\pi}{6}\right) = 1$

$x + \frac{\pi}{6} = \frac{\pi}{2}, \ldots$

$\therefore x = \frac{\pi}{3}$ (2)

$\therefore$ A is point $\left(\frac{\pi}{3}, 2\right)$

B is point where curve cuts x axis ie $y = 0$

$\therefore \sin\left(x + \frac{\pi}{6}\right) = 0$

$x + \frac{\pi}{6} = 0, \pi, \ldots$

$\therefore x = -\frac{\pi}{6}, \frac{5\pi}{6}, \ldots$ (2)

$\therefore B$ is $\left(\frac{5\pi}{6}, 0\right)$

(c) $y_1 = \frac{3}{2}x + \frac{1}{2}$

$y_2 = -5x + 7$

$m_1 = \frac{3}{2} \qquad m_2 = -5$

$\therefore \tan\theta = \left|\frac{\frac{3}{2} + 5}{1 + (\frac{3}{2} \times -5)}\right|$ 2

$= 1$

$\therefore \theta = 45^\circ$

Q3

(a)(i) LHS $\dfrac{1-\cos 2x}{1+\cos 2x}$

$= \dfrac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}$

$= \dfrac{2\sin^2 x}{2\cos^2 x}$

$= \tan^2 x$

$= \sec^2 x - 1$ (3)

$=$ RHS

(ii) $\displaystyle\int_0^{\pi/4} \frac{1-\cos 2x}{1+\cos 2x}\,dx$

$\displaystyle= \int_0^{\pi/4} (\sec^2 x - 1)\,dx$

$= \left[\tan x - x\right]_0^{\pi/4}$

$= \tan \pi/4 - \pi/4$ (3)

$= 1 - \pi/4$

(b) (i) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi h^3$ $(r = h)$

$\dfrac{dV}{dh} = \pi h^2$

and $\dfrac{dV}{dt} = 8$ 3

$\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$

$= \dfrac{1}{\pi h^2} \times 8$

when $h = 10$, $\dfrac{dh}{dt} = \dfrac{1}{100\pi} \times 8$

$\boxed{= \dfrac{2}{25\pi}}$

∴ h increasing at $\frac{2}{25\pi}$ m/min.

(ii) $A = \pi r^2 = \pi h^2$

$\dfrac{dA}{dh} = 2\pi h$

$\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$

$= 2\pi h \times \dfrac{2}{25\pi}$

when $h = 10$, $\dfrac{dA}{dt} = 20\pi \times \dfrac{2}{25\pi}$

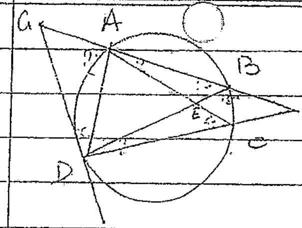
$= \dfrac{40}{25}$ 3

$= \dfrac{8}{5}$

∴ area of bases ↑ at rate of $8/25$ m^2/min.

Q4

(a)



(ii) $\angle ACD = \angle ADG = x^\circ$ (∠ bet tan + ch)

∴ $\angle ABD = x$ (∠s subt by same arc)

∴ $\angle FBE = \angle FCE = 180^\circ - x$ (supp ∠s)

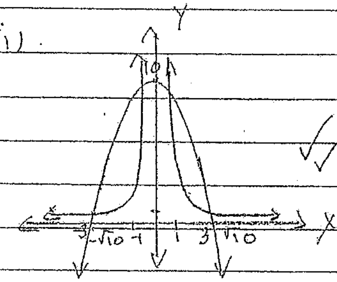
∴ $\angle BEC + \angle BFC = 360^\circ - 2(180^\circ - x)$

$= 2x$ (∠s sum quad)

∴ $\angle BEC + \angle BFC = 2\angle ADG$. (4)

(iv)

b (i)



3

(ii) By substitution

$x = 1$, $y = 9$

∴ intersect at $x = 1$

$x = 3$ $y = 1$

∴ intersect at $x = 3$ 1

By symm $x = -1$, $x = -3$, will also be pt. of intes.

OR solve

$\dfrac{9}{x^2} = 10 - x^2$ or

$x^4 - 10x^2 + 9 = 0$

$(x^2 - 9)(x^2 - 1) = 0$

$x = \pm 3 \ \pm 1$

iii) from graph.

$10 - x^2 > 9/x^2$

for $-3 < x < -1$, (2)

$1 < x < 3$

Q5

(a) Test n=1

$7^1 - 1 = 6$

∴ true for n=1

Assume stmt true for k

ie $\frac{7^k - 1}{3} = M$ (where M is $\mathbb{Z}$)

ie $7^k - 1 = 3M$

Prove stmt true for k+1

ie $7^{k+1} - 1$

$= 7.7^k - 1 = 7.7^k - 7 + 6$

$= 7(7^k - 1) + 6$

$= 7(3M) + 6$

$= 3(7M) + 6$

$= 3(7M + 2)$

which is divisible by 3 ∴ stmt is true for n = k + for n = k+1. 4.

∴ true for all n.

(b) $f(x) = \sin^{-1}x + \cos^{-1}x$

(i) $f(0) = \sin^{-1}0 + \cos^{-1}0$

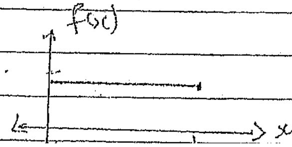
$= \pi/2$

(ii) $f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$

$= 0$

(iii) $f'(x) = 0$ for all x.

∴ graph is horiz line thro' $\pi/2$



(c) $\frac{x^2 - 1}{x^2 + 1}$

$= \frac{(\tan\theta + \sec\theta)^2 - 1}{(\tan\theta + \sec\theta)^2 + 1}$

$= \frac{\left(\frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2}\right)^2 - 1}{\left(\frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2}\right)^2 + 1}$

$= \left(\frac{2t + t^2 + 1}{1 - t^2}\right)^2 - 1 + t^2$

Q6 (i) $U_1 = 50000(1.01)^n - M$

(a) (ii) $M = \frac{PR^N(R-1)}{R^N - 1}$

$= \frac{50000(1.01)^{240}(0.01)}{1.01^{240} - 1}$

$= \$550.54$ 2

(ii) The amt owing would increase & loan won't be paid off. (1)

(iii) $M = \frac{PR^N(R-1)}{R^N - 1}$

$MR^N - M = PRR^N - PR^N$

ie $R^N(M - PR + P) = M$

$R^N = \frac{M}{M - PR + P}$

Want $M = 2 \times 550.54$

$P = 50000$

$R = 1.01$

∴ $1.01^N = \frac{2 \times 550.54}{2 \times 550.54 - 50000 \times 1.01 + 50000}$

$= 1.832$ (3)

∴ N = 60.8 mths.

(b)

(i) Midpt $M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left[a(p+q), \frac{a(p^2+q^2)}{2}\right]$

M & T have same abscissa, ∴ MT ∥ Y axis (axis of parab (3)

(ii) $x^2 = 4ay$

$4ay = \frac{a^2(p+q)^2}{4}$ (sub x

∴ N is $\left[a(p+q), \frac{a(p+q)^2}{4}\right]$

midpt of MT $= \left[a(p+q), \frac{\frac{a(p^2+q^2)}{2} + \dots}{2}\right]$

$= \left(a(p+q), \frac{a(p+q)^2}{4}\right)$

~~= coord of N~~ (3)

∴ N is midpt. of MT

(iii) $y = \frac{x^2}{4a}$

$y' = \frac{x}{2a}$

m at N $= \frac{a(p+q)}{2a}$

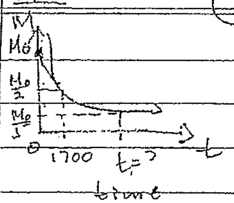
$= \frac{p+q}{2}$ (2)

$= m_{PQ}$

∴ PQ ∥ tan at N

Q7 (12)

(a)



$M = M_0 e^{-kt}$

when $t = 0$, $M = M_0$

$\therefore M_0$ rep orig mass.

$t = 1700$, $M = \frac{M_0}{2}$

$\therefore \frac{M_0}{2} = M_0 e^{-k1700}$

ie $0.5 = e^{-1700k}$

$k = 0.0004077$

$\therefore M = M_0 e^{-0.00004077t}$

Need t when only 20% of M_0 remains

ie $\frac{M_0}{5} = M_0 e^{-0.0004077t}$

ie $0.2 = e^{-0.0004077t}$ (5)

$\therefore t = 3900$ yrs

(to nearest 100yrs)

(b) $x = vt\cos\alpha$

$y = vt\sin\alpha - \frac{1}{2}gt^2$

horizontal range

$\Rightarrow y = 0$

when $y = 0$, $vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$t(v\sin\alpha - \frac{1}{2}gt) = 0$

$\therefore t = 0$ (initial condition) $\quad t = \frac{2V\sin\alpha}{g}$

when $t = \frac{2V\sin\alpha}{g}$

$x = V\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$

$= \frac{2V^2\sin\alpha\cos\alpha}{g}$ (4)

$\therefore$ horizontal range $= \frac{V^2\sin 2\alpha}{g}$

(c) To achieve max range, Vijay should hit ball at 45°

$\therefore$ horizontal range $= \frac{50^2\sin 90°}{10}$

$= 250$ m.

but this will land in lake

$\therefore$ Vijay should try to hit the ball so that horizontal range is ≤ 239 m. (3)